

# A Note on Approximately Divisible $C^*$ -algebras

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**Abstract:** Let  $\mathcal{A}$  be a separable, unital, approximately divisible  $C^*$ -algebra. We show that  $\mathcal{A}$  is generated by two self-adjoint elements and the topological free entropy dimension of any finite generating set of  $\mathcal{A}$  is less than or equal to 1. In addition, we show that the similarity degree of  $\mathcal{A}$  is at most 5. Thus an approximately divisible  $C^*$ -algebra has an affirmative answer to Kadison's similarity problem.

**Keywords:** Approximately divisible  $C^*$ -algebra, generators, topological free entropy dimension, similarity degree

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## 1. Introduction

The class of approximately divisible  $C^*$ -algebras was introduced by B. Blackadar, A. Kumjian and M. Rørdam in [1], where they constructed a large class of simple  $C^*$ -algebras having trivial non-stable  $K$ -theory. They showed that the class of approximately divisible  $C^*$ -algebras contains all simple unital AF-algebras and most of the simple unital AH-algebras with real rank 0, and every nonrational noncommutative torus.

The theory of free entropy and free entropy dimension was developed by D. Voiculescu in the 1990's. It has been a very powerful tool in the recent study of finite von Neumann algebras. In [17], D. Voiculescu introduced the notion of topological free entropy dimension of elements in a unital  $C^*$ -algebra as an analogue of free entropy dimension in the context of  $C^*$ -algebra. Recently, D. Hadwin and J. Shen [7] obtained some interesting results on topological free entropy dimensions of unital  $C^*$ -algebras, which includes the irrational rotation  $C^*$ -algebras, UHF algebras and minimal tensor products of reduced free group  $C^*$ -algebras. Thus it will be interesting to consider the topological free entropy dimensions for larger class of unital  $C^*$ -algebras. One motivation of the paper is to calculate the topological free entropy dimensions in the approximately divisible unital  $C^*$ -algebras.

Note that Voiculescu's topological free entropy dimension is defined for the finitely generated  $C^*$ -algebras. Therefore it is natural to consider the generator problem for approximately divisible unital  $C^*$ -algebras before we carry out the calculation of the topological free entropy for approximately divisible unital  $C^*$ -algebras. In fact the generator

problem for  $C^*$ -algebras and the one for von Neumann algebras have been studied by many people and many results have been obtained. For example, C. Olsen and W. Zame [9] showed that if  $\mathcal{A}$  is a unital separable  $C^*$ -algebra and  $\mathcal{B}$  is a UHF algebra, then  $\mathcal{A} \otimes \mathcal{B}$  is generated by two self-adjoint elements in  $\mathcal{A} \otimes \mathcal{B}$ . It is clear that such  $\mathcal{A} \otimes \mathcal{B}$  is approximately divisible. In the paper we obtain the following result (see Theorem 3.1), which is an extension of C. Olsen and W. Zame's result in [9].

**Theorem:** If  $\mathcal{A}$  is a unital separable approximately divisible  $C^*$ -algebra, then  $\mathcal{A}$  is generated by two self-adjoint elements in  $\mathcal{A}$ , i.e.,  $\mathcal{A}$  is singly generated.

Next, we develop the techniques from [7] and compute the topological free entropy dimension of any finite family of self-adjoint generators of a unital separable approximately divisible  $C^*$ -algebra. More specifically, we obtain the following result (see Theorem 4.3).

**Theorem:** Let  $\mathcal{A}$  be a unital separable approximately divisible  $C^*$ -algebra. If  $\mathcal{A}$  has approximation property, then

$$\delta_{top}(x_1, \dots, x_n) = 1,$$

where  $x_1, \dots, x_n$  is any family of self-adjoint generators of  $\mathcal{A}$ .

In the last part of the paper, we study the Kadison's similarity problem for approximately divisible  $C^*$ -algebras. In [8], R. Kadison formulated his famous similarity problem for a  $C^*$ -algebra  $\mathcal{A}$ , which asks the following question: Let  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  ( $\mathcal{H}$  is a Hilbert space) be a unital bounded homomorphism. Is  $\pi$  similar to a  $*$ -homomorphism, that is, there exists an invertible operator  $S \in \mathcal{B}(\mathcal{H})$  such that  $S^{-1}\pi(\cdot)S$  is a  $*$ -homomorphism?

G. Pisier [10] introduced a powerful concept, similarity degree of a  $C^*$ -algebra, to determine whether Kadison's similarity problem for a  $C^*$ -algebra has an affirmative answer. In fact, he showed that a  $C^*$ -algebra  $\mathcal{A}$  has an affirmative answer to Kadison's similarity problem if and only if the similarity degree of  $\mathcal{A}$ ,  $d(\mathcal{A})$ , is finite. The similarity degrees of some classes of  $C^*$ -algebras have been known, which we list as below.

- (1)  $\mathcal{A}$  is nuclear if and only if  $d(\mathcal{A}) = 2$  ([2], [3], [13]);
- (2) if  $\mathcal{A} = \mathcal{B}(\mathcal{H})$ , then  $d(\mathcal{A}) = 3$  ([12]);
- (3)  $d(\mathcal{A} \otimes \mathcal{K}(\mathcal{H})) \leq 3$  for any  $C^*$ -algebra  $\mathcal{A}$  ([6], [11]);
- (4) if  $\mathcal{M}$  is a factor of type  $II_1$  with property  $\Gamma$ , then  $d(\mathcal{M}) = 3$  ([4]).

The last result (see Theorem 5.1) we obtain in the paper is the calculation of similarity degree of approximately divisible  $C^*$ -algebras.

**Theorem:** If  $\mathcal{A}$  is a unital separable approximately divisible  $C^*$ -algebra, then

$$d(\mathcal{A}) \leq 5.$$

As a corollary, an approximately divisible  $C^*$ -algebra has an affirmative answer to Kadison's similarity problem.

The paper has five sections. In section 2, we recall the definition of approximately divisible  $C^*$ -algebra. The generator problem for an approximately divisible  $C^*$ -algebra is considered in section 3. The computation of topological free entropy dimension in an

approximately divisible  $C^*$ -algebra is carried out in section 4. In section 5, we consider the similarity degree of an approximately divisible  $C^*$ -algebra.

## 2. Notation and preliminaries

In this section, we will introduce some notation that will be needed later and recall the definition of approximately divisible  $C^*$ -algebra introduced by B. Blackadar, A. Kumjian and M. Rørdam [1].

Let  $\mathcal{M}_k(\mathbb{C})$  be the  $k \times k$  full matrix algebra with entries in  $\mathbb{C}$ , and  $\mathcal{M}_k^{s.a.}(\mathbb{C})$  be the subalgebra of  $\mathcal{M}_k(\mathbb{C})$  consisting of all self-adjoint matrices of  $\mathcal{M}_k(\mathbb{C})$ . Let  $\mathcal{U}_k$  be the group of all unitary matrices in  $\mathcal{M}_k(\mathbb{C})$ . Let  $\mathcal{M}_k(\mathbb{C})^n$  denote the direct sum of  $n$  copies of  $\mathcal{M}_k(\mathbb{C})$ . Let  $(\mathcal{M}_k^{s.a.}(\mathbb{C}))^n$  be the direct sum of  $n$  copies of  $\mathcal{M}_k^{s.a.}(\mathbb{C})$ .

The following lemma is a well-known fact.

LEMMA 2.1. *Suppose  $\mathcal{B}$  is a finite-dimensional  $C^*$ -algebra. Then there exist positive integers  $r$  and  $k_1, \dots, k_r$  such that*

$$\mathcal{B} \cong \mathcal{M}_{k_1}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{k_r}(\mathbb{C}).$$

DEFINITION 2.1. *Suppose*

$$\mathcal{B} \cong \mathcal{M}_{k_1}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{k_r}(\mathbb{C})$$

*is a finite-dimensional  $C^*$ -algebra for some positive integers  $r, k_1, \dots, k_r$ . Define the rank of  $\mathcal{B}$  to be*

$$\text{Rank}(\mathcal{B}) = k_1 + \dots + k_r,$$

*the subrank of  $\mathcal{B}$  to be*

$$\text{SubRank}(\mathcal{B}) = \min\{k_1, \dots, k_r\}.$$

The following definition is Definition 1.2 in [1].

DEFINITION 2.2. *A separable unital  $C^*$ -algebra  $\mathcal{A}$  with the unit  $I_{\mathcal{A}}$  is approximately divisible if, for every  $x_1, \dots, x_n \in \mathcal{A}$  and  $\varepsilon > 0$ , there is a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}$  such that*

- (1)  $I_{\mathcal{A}} \in \mathcal{B}$ ;
- (2)  $\text{SubRank}(\mathcal{B}) \geq 2$ ;
- (3)  $\|x_i y - y x_i\| < \varepsilon$  for  $i = 1, \dots, n$  and all  $y \in \mathcal{B}$  with  $\|y\| \leq 1$ .

The following proposition is also taken from Theorem 1.3 and Corollary 2.10 in [1].

PROPOSITION 2.1. ([1]) *Let  $\mathcal{A}$  be a unital separable approximately divisible  $C^*$ -algebra with the unit  $I_{\mathcal{A}}$ . Then there exists an increasing sequence  $\{\mathcal{A}_m\}_{m=1}^{\infty}$  of subalgebras of  $\mathcal{A}$  such that*

- (1)  $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$ ,
- (2) for any positive integer  $m$ ,  $\mathcal{A}'_m \cap \mathcal{A}_{m+1}$  contains a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  with  $I_{\mathcal{A}} \in \mathcal{B}$  and  $\text{SubRank}(\mathcal{B}) \geq 2$ ,
- (3) for any positive integers  $m$  and  $k$ , there is a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}'_m \cap \mathcal{A}$  with  $I_{\mathcal{A}} \in \mathcal{B}$  and  $\text{SubRank}(\mathcal{B}) \geq k$ .

### 3. Generator problem of approximately divisible $C^*$ -algebras

In this section we prove that every unital separable approximately divisible  $C^*$ -algebra is singly generated, i.e., generated by two self-adjoint elements.

**THEOREM 3.1.** *If  $\mathcal{A}$  is a unital separable approximately divisible  $C^*$ -algebra, then  $\mathcal{A}$  is singly generated.*

**PROOF.** Since  $\mathcal{A}$  is separable, there exists a sequence of self-adjoint elements  $\{x_i\}_{i=1}^\infty \subset \mathcal{A}$  that generate  $\mathcal{A}$  as a  $C^*$ -algebra.

**CLAIM 3.1.** *There exists a sequence of finite-dimensional subalgebras  $\{\mathcal{B}_n\}_{n=1}^\infty$  of  $\mathcal{A}$  so that the following hold:*

- (1)  $\forall n \in \mathbb{N}$ ,  $I_{\mathcal{A}} \in \mathcal{B}_n$ , where  $I_{\mathcal{A}}$  is the unit of  $\mathcal{A}$ ;
- (2)  $\text{SubRank}(\mathcal{B}_1) \geq 3$ , and for any  $n \geq 2$ ,

$$\text{SubRank}(\mathcal{B}_n) \geq n \cdot (\text{Rank}(\mathcal{B}_1))^2 \cdots (\text{Rank}(\mathcal{B}_{n-1}))^2 + 3;$$

- (3) if  $n \neq m$ , then  $\mathcal{B}_n$  commutes with  $\mathcal{B}_m$ ;
- (4) for any  $n \in \mathbb{N}$ ,

$$\text{dist}(x_p, \mathcal{B}'_n \cap \mathcal{A}) < 2^{-n}, \quad \forall 1 \leq p \leq n,$$

where  $\text{dist}(x_p, \mathcal{B}'_n \cap \mathcal{A}) = \inf\{\|x_p - y\| : y \in \mathcal{B}'_n \cap \mathcal{A}\}$ .

*Proof of the claim.* It follows from Proposition 2.1 that there exists an increasing sequence  $\{\mathcal{A}_m\}_{m=1}^\infty$  of subalgebras of  $\mathcal{A}$  such that

- (a)  $\mathcal{A} = \overline{\bigcup_m \mathcal{A}_m}^{\|\cdot\|}$ ,
- (b) for any positive integer  $m$ ,  $\mathcal{A}'_m \cap \mathcal{A}_{m+1}$  contains a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  with  $I_{\mathcal{A}} \in \mathcal{B}$  and  $\text{SubRank}(\mathcal{B}) \geq 2$ ,
- (c) for any positive integers  $m$  and  $k$ , there is a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}'_m \cap \mathcal{A}$  with  $I_{\mathcal{A}} \in \mathcal{B}$  and  $\text{SubRank}(\mathcal{B}) \geq k$ .

Instead of proving Claim 3.1 directly, we will prove a stronger result by replacing the statement (3) in Claim 3.1 with the following one:

(3') there exist two increasing sequences  $\{s_n\}_{n=1}^\infty$  and  $\{t_n\}_{n=1}^\infty$  of positive integers such that, for any  $n \in \mathbb{N}$ ,  $s_n \leq t_n \leq s_{n+1}$  and  $\mathcal{B}_n \subseteq \mathcal{A}'_{s_n} \cap \mathcal{A}_{t_n}$ .

We prove this stronger claim by using the induction on  $n$ .

**Base step:** Note that  $\mathcal{A} = \overline{\bigcup_m \mathcal{A}_m}^{\|\cdot\|}$ . For  $x_1 \in \mathcal{A}$ , there are a positive integer  $s_1$  and a self-adjoint element  $y_1^{(1)} \in \mathcal{A}_{s_1}$  such that  $\|x_1 - y_1^{(1)}\| < \frac{1}{2}$ . By the restriction (b) on the subalgebras  $\{\mathcal{A}_m\}_{m=1}^\infty$ , we know that there exist two finite-dimensional subalgebras  $\mathcal{C}_{s_1+1}, \mathcal{C}_{s_1+2}$  in  $\mathcal{A}$  such that,

- (i)  $I_{\mathcal{A}} \in \mathcal{C}_{s_1+1}$  and  $I_{\mathcal{A}} \in \mathcal{C}_{s_1+2}$ ;
- (ii)  $\mathcal{C}_{s_1+1} \subseteq \mathcal{A}'_{s_1} \cap \mathcal{A}_{s_1+1}$  and  $\mathcal{C}_{s_1+2} \subseteq \mathcal{A}'_{s_1+1} \cap \mathcal{A}_{s_1+2}$ ;
- (iii)  $\text{SubRank}(\mathcal{C}_{s_1+1})$  and  $\text{SubRank}(\mathcal{C}_{s_1+2})$  are at least 2.

Let  $t_1 = s_1 + 2$ ,  $\mathcal{B}_1 = C^*(\mathcal{C}_{s_1+1}, \mathcal{C}_{t_1})$  the  $*$ -subalgebra generated by  $\mathcal{C}_{s_1+1}$  and  $\mathcal{C}_{t_1}$  in  $\mathcal{A}$ . Then  $\text{SubRank}(\mathcal{B}_1) \geq 3$  and  $\mathcal{B}_1 \subseteq \mathcal{A}'_{s_1} \cap \mathcal{A}_{t_1}$ .

**Inductive step:** Now suppose the stronger claim is true when  $n \leq k-1$ , i.e., there exists a family of finite-dimensional C\*-algebras  $\{\mathcal{B}_n\}_{n=1}^{k-1}$  of  $\mathcal{A}$ , and two increasing sequences of positive integers  $\{s_n\}_{n=1}^{k-1}$  and  $\{t_n\}_{n=1}^{k-1}$  that satisfy (1), (2), (3') and (4).

For  $x_1, \dots, x_k$  in  $\mathcal{A}$ , from the restriction (a) on  $\{\mathcal{A}_m\}_{m=1}^\infty \subseteq \mathcal{A}$ , we know that there are a positive integer  $s_k$  with  $s_k \geq t_{k-1}$  and self-adjoint elements  $y_1^{(k)}, \dots, y_k^{(k)}$  in  $\mathcal{A}_{s_k}$  such that  $\|x_i - y_i^{(k)}\| < 2^{-k}$  for  $1 \leq i \leq k$ . From the restriction (b) on  $\{\mathcal{A}_m\}_{m=1}^\infty \subseteq \mathcal{A}$ , there exists a family  $\{\mathcal{C}_{s_k+1}, \mathcal{C}_{s_k+2}, \dots\}$  of finite-dimensional subalgebras in  $\mathcal{A}$  such that,

- (i)  $I_{\mathcal{A}} \in \mathcal{C}_{s_k+i}, \forall i \geq 1$ ;
- (ii)  $\mathcal{C}_{s_k+i} \subseteq \mathcal{A}'_{s_k+i-1} \cap \mathcal{A}_{s_k+i}, \forall i \geq 1$ ;
- (iii)  $\text{SubRank}(\mathcal{C}_{s_k+i}) \geq 2, \forall i \geq 1$ .

By (ii), we know  $\{\mathcal{C}_{s_k+1}, \mathcal{C}_{s_k+2}, \dots\}$  is a commuting sequence of subalgebras of  $\mathcal{A}$ . Combining with (iii), we get that there is a positive integer  $t_k$  such that

$$\text{SubRank}(C^*(\mathcal{C}_{s_k+1}, \dots, \mathcal{C}_{t_k})) \geq k \cdot (\text{Rank}(\mathcal{B}_1))^2 \cdots (\text{Rank}(\mathcal{B}_{k-1}))^2 + 3,$$

where  $C^*(\mathcal{C}_{s_k+1}, \dots, \mathcal{C}_{t_k})$  is the C\*-subalgebra generated by  $\mathcal{C}_{s_k+1}, \dots, \mathcal{C}_{t_k}$  in  $\mathcal{A}$ . Moreover,  $I_{\mathcal{A}} \in C^*(\mathcal{C}_{s_k+1}, \dots, \mathcal{C}_{t_k})$  is a finite-dimensional C\*-subalgebra in  $\mathcal{A}'_{s_k} \cap \mathcal{A}_{t_k}$ . Let

$$\mathcal{B}_k = C^*(\mathcal{C}_{s_k+1}, \dots, \mathcal{C}_{t_k})$$

and it is not hard to check that  $\mathcal{B}_1, \dots, \mathcal{B}_k$  satisfy the conditions (1), (2), (3') and (4) in the stronger claim. *This completes the proof of the claim.*

Let  $\{\mathcal{B}_n\}_{n=1}^\infty$  be as in Claim 3.1. For any positive integer  $n$ , since  $\mathcal{B}_n$  is a finite-dimensional C\*-algebra, there exist positive integers  $r_n$  and  $k_1^{(n)}, \dots, k_{r_n}^{(n)}$  such that

$$\mathcal{B}_n \cong \mathcal{M}_{k_1^{(n)}}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_{r_n}^{(n)}}(\mathbb{C}).$$

Let  $\{e_{ij}^{(n,s)} : 1 \leq i, j \leq k_s^{(n)}\}$  be the canonical system of matrix units for  $\mathcal{M}_{k_s^{(n)}}(\mathbb{C})$ . If there is no confusion arising, we can further assume that  $\{e_{ij}^{(n,s)} : 1 \leq i, j \leq k_s^{(n)}, 1 \leq s \leq r_n\}$  consists a system of matrix units of  $\mathcal{B}_n$ . Note that  $\mathcal{B}_n$  contains the unit  $I_{\mathcal{A}}$  of  $\mathcal{A}$ , so

$$\sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)}} e_{ii}^{(n,s)} = I_{\mathcal{A}}.$$

Define

$$p_n = \sum_{s=1}^{r_n} e_{k_s^{(n)}, k_s^{(n)}}^{(n,s)} \quad \text{for } n \geq 1. \quad (1)$$

Then  $p_n$  is a projection of  $\mathcal{B}_n$ . It is clear that

$$p_n e_{11}^{(n,s)} = 0 \quad \text{for } 1 \leq s \leq r_n. \quad (2)$$

**CLAIM 3.2.** *Let  $\{x_n\}_{n=1}^\infty, \{\mathcal{B}_n\}_{n=1}^\infty, \{r_n\}_{n=1}^\infty$  and  $\{p_n\}_{n=1}^\infty$  be defined as above. For any positive integer  $n$ , there exists  $z_n = z_n^* \in \mathcal{A}$  with  $\|z_n\| = 2^{-(r_1 + \cdots + r_n + 1)}$  so that*

$$(i) (I_{\mathcal{A}} - p_n)p_{n-1} \cdots p_1 \cdot z_n \cdot p_1 \cdots p_{n-1}(I_{\mathcal{A}} - p_n) = z_n,$$

(ii)  $\text{dist}(x_j, C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)) < 2^{-n}$  for  $1 \leq j \leq n$ , where  $C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)$  is the  $C^*$ -subalgebra generated by  $\mathcal{B}_1, \dots, \mathcal{B}_n, z_n$  in  $\mathcal{A}$ .

*Proof of the claim.* By Claim 3.1, for any positive integer  $n$  and  $x_1, \dots, x_n$ , we know

$$\text{dist}(x_j, \mathcal{B}'_n \cap \mathcal{A}) < 2^{-n} \quad \text{for } 1 \leq j \leq n.$$

Thus there exist self-adjoint elements  $y_1^{(n)}, \dots, y_n^{(n)}$  in  $\mathcal{A}$  that commute with  $\mathcal{B}_n$  and

$$\|x_j - y_j^{(n)}\| < 2^{-n} \quad \text{for } 1 \leq j \leq n.$$

Let

$$z_1 = \frac{1}{2^{1+r_1}} \cdot \frac{\sum_{s=1}^{r_1} e_{22}^{(1,s)} y_1^{(1)}}{\left\| \sum_{s=1}^{r_1} e_{22}^{(1,s)} y_1^{(1)} \right\|}. \quad (3)$$

With  $\text{SubRank}(\mathcal{B}_1) \geq 3$ , we have

$$(I_{\mathcal{A}} - p_1) \cdot z_1 \cdot (I_{\mathcal{A}} - p_1) = z_1.$$

By the equation (3) and the fact that  $y_1^{(1)}$  commutes with  $\mathcal{B}_1$ , we know

$$y_1^{(1)} = \left( 2^{1+r_1} \cdot \left\| \sum_{s=1}^{r_1} e_{22}^{(1,s)} y_1^{(1)} \right\| \right) \cdot \left( \sum_{s=1}^{r_1} \sum_{i=1}^{k_s^{(1)}} e_{i,2}^{(1,s)} \cdot z_1 \cdot e_{2,i}^{(1,s)} \right).$$

Thus we know that  $y_1^{(1)}$  is in the  $C^*$ -algebra generated by  $\mathcal{B}_1$  and  $z_1$ , whence

$$\text{dist}(x_1, C^*(\mathcal{B}_1, z_1)) \leq \text{dist}(x_1, y_1^{(1)}) < 2^{-1}.$$

Now let us construct  $z_n$  for any positive integer  $n \geq 2$ . Let

$$\begin{aligned} \Delta_{n-1} &= \{(i_1, s_1) \times (j_1, t_1) \times \dots \times (i_{n-1}, s_{n-1}) \times (j_{n-1}, t_{n-1}) : \\ &\quad 1 \leq i_1 \leq k_{s_1}^{(1)}, 1 \leq j_1 \leq k_{t_1}^{(1)}, 1 \leq s_1, t_1 \leq r_1, \\ &\quad \dots, 1 \leq i_{n-1} \leq k_{s_{n-1}}^{(n-1)}, 1 \leq j_{n-1} \leq k_{t_{n-1}}^{(n-1)}, 1 \leq s_{n-1}, t_{n-1} \leq r_{n-1}\}. \end{aligned}$$

It is not hard to check that the cardinality of the set  $\Delta_{n-1}$  satisfies

$$\text{Card}(\Delta_{n-1}) = \prod_{i=1}^{n-1} (\text{Rank}(\mathcal{B}_i))^2.$$

Hence, for any  $1 \leq j \leq n$ , there is a one-to-one mapping  $f_j^{(n)}$  from the index set  $\Delta_{n-1}$  onto the set

$$\{i \in \mathbb{N} \mid (j-1) \cdot \text{Card}(\Delta_{n-1}) + 2 \leq i \leq j \cdot \text{Card}(\Delta_{n-1}) + 1\}.$$

For any index

$$\alpha = (i_1, s_1) \times (j_1, t_1) \times \dots \times (i_{n-1}, s_{n-1}) \times (j_{n-1}, t_{n-1}) \in \Delta_{n-1}$$

and any  $1 \leq j \leq n$ , we should define

$$\alpha(y_j^{(n)}) = e_{k_{s_{n-1}}^{(n-1)}, i_{n-1}}^{(n-1, s_{n-1})} \dots e_{k_{s_1}^{(1)}, i_1}^{(1, s_1)} \cdot y_j^{(n)} \cdot e_{j_1, k_{t_1}^{(1)}}^{(1, t_1)} \dots e_{j_{n-1}, k_{t_{n-1}}^{(n-1)}}^{(n-1, t_{n-1})} \in \mathcal{A}. \quad (4)$$

By Claim 3.1, we know that  $\text{SubRank}(\mathcal{B}_n) \geq n \cdot \text{Card}(\Delta_{n-1}) + 3$ . It follows that

$$z_n = c_n \cdot \sum_{s=1}^{r_n} \sum_{j=1}^n \sum_{\alpha \in \Delta_{n-1}} \left( e_{f_j^{(n)}(\alpha), f_j^{(n)}(\alpha)+1}^{(n,s)} \cdot \alpha(y_j^{(n)}) + (e_{f_j^{(n)}(\alpha), f_j^{(n)}(\alpha)+1}^{(n,s)} \cdot \alpha(y_j^{(n)}))^* \right) \quad (5)$$

is well defined and contained in  $\mathcal{A}$ , where  $c_n$  is a constant such that

$$\|z_n\| = 2^{-(r_1+\dots+r_n+1)}. \quad (6)$$

From the construction of  $z_n$ , it follows that  $z_n = z_n^*$  and

$$z_n = (I_{\mathcal{A}} - p_n) \cdot p_{n-1} \cdots p_1 \cdot z_n \cdot p_1 \cdots p_{n-1} \cdot (I_{\mathcal{A}} - p_n).$$

To prove  $\text{dist}(x_j, C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)) < 2^{-n}$  for  $1 \leq j \leq n$ , it is sufficient to prove that  $\{y_1^{(n)}, \dots, y_n^{(n)}\} \subseteq C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)$ . Because  $\mathcal{B}_n$  commutes with  $y_1^{(n)}, \dots, y_n^{(n)}$ , for any  $\alpha \in \Delta_{n-1}$  and  $1 \leq j \leq n$ , from the equation (5) it follows that

$$\alpha(y_j^{(n)}) = \sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)}} e_{i, f_j^{(n)}(\alpha)}^{(n,s)} \cdot \left( \frac{1}{c_n} z_n \right) \cdot e_{f_j^{(n)}(\alpha)+1, i}^{(n,s)}.$$

This implies that  $\alpha(y_j^{(n)}) \in C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)$ .

Suppose  $\alpha = (i_1, s_1) \times (j_1, t_1) \times \dots \times (i_{n-1}, s_{n-1}) \times (j_{n-1}, t_{n-1}) \in \Delta_{n-1}$ . Again because  $\mathcal{B}_1, \dots, \mathcal{B}_{n-1}$  are commuting, for any  $1 \leq j \leq n$ , from the equation (4) it follows that

$$\begin{aligned} & e_{i_{n-1}, i_{n-1}}^{(n-1, s_{n-1})} \cdots e_{i_1, i_1}^{(1, s_1)} \cdot y_j^{(n)} \cdot e_{j_1, j_1}^{(1, t_1)} \cdots e_{j_{n-1}, j_{n-1}}^{(n-1, t_{n-1})} \\ &= e_{i_{n-1}, k_{s_{n-1}}^{(n-1)}}^{(n-1, s_{n-1})} \cdots e_{i_1, k_{s_1}^{(1)}}^{(1, s_1)} \cdot \alpha(y_j^{(n)}) \cdot e_{k_{t_1}^{(1)}, j_1}^{(1, t_1)} \cdots e_{k_{t_{n-1}}^{(n-1)}, j_{n-1}}^{(n-1, t_{n-1})} \end{aligned}$$

and

$$\begin{aligned} y_j^{(n)} &= \sum_{s_1, t_1=1}^{r_1} \sum_{i_1=1}^{k_{s_1}^{(1)}} \sum_{j_1=1}^{k_{t_1}^{(1)}} \cdots \sum_{s_{n-1}, t_{n-1}=1}^{r_{n-1}} \sum_{i_{n-1}=1}^{k_{s_{n-1}}^{(n-1)}} \sum_{j_{n-1}=1}^{k_{t_{n-1}}^{(n-1)}} \\ & \quad e_{i_{n-1}, i_{n-1}}^{(n-1, s_{n-1})} \cdots e_{i_1, i_1}^{(1, s_1)} \cdot y_j^{(n)} \cdot e_{j_1, j_1}^{(1, t_1)} \cdots e_{j_{n-1}, j_{n-1}}^{(n-1, t_{n-1})}. \end{aligned}$$

Thus  $y_j^{(n)} \in C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)$ . Hence

$$\text{dist}(x_j, C^*(\mathcal{B}_1, \dots, \mathcal{B}_n, z_n)) \leq \text{dist}(x_j, y_j^{(n)}) < 2^{-n} \quad \text{for } 1 \leq j \leq n.$$

*This completes the proof of the claim.*

Let  $\{x_n\}_{n=1}^\infty$ ,  $\{\mathcal{B}_n\}_{n=1}^\infty$ ,  $\{r_n\}_{n=1}^\infty$ ,  $\{p_n\}_{n=1}^\infty$  and  $\{z_n\}_{n=1}^\infty$  be as above. From the equation (2), the fact (i) of Claim 3.2 and the construction of  $z_n$ , we can get some basic facts of  $z_n$ . Let us list them below:

- (F1)  $p_n z_n = z_n p_n = 0$ ,
- (F2)  $z_n \cdot e_{11}^{(m,s)} = e_{11}^{(m,s)} \cdot z_n = 0$  for  $m \leq n$  and  $1 \leq s \leq r_m$ ,
- (F3)  $z_n \cdot z_m = 0$  for any  $n \neq m$ .

Let  $p_0 = I_{\mathcal{A}}$  and  $r_0 = 0$ . For any  $n \geq 1$ , let

$$a_n = p_1 \cdots p_{n-1} \sum_{s=1}^{r_n} 2^{-r_1 - \cdots - r_{n-1} - s} \cdot e_{11}^{(n,s)} + z_n, \quad (7)$$

$$b_n = 2^{-2n} p_1 \cdots p_{n-1} \sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)} - 1} (e_{i,i+1}^{(n,s)} + e_{i+1,i}^{(n,s)}). \quad (8)$$

From the proven facts (F1), (F2) and (F3), we have

$$a_n \cdot a_m = 0. \quad \text{for } n \neq m \quad (9)$$

Combining the fact (F2), the equation (6) and the fact  $e_{11}^{(n,s)} \cdot e_{11}^{(n,s_1)} = e_{11}^{(n,s_1)} \cdot e_{11}^{(n,s)} = 0$  ( $s \neq s_1$ ), it is clear that

$$\|a_n\| = \max\{\|p_1 \cdots p_{n-1} \cdot 2^{-r_1 - \cdots - r_{n-1} - s} \cdot e_{11}^{(n,s)}\|_{s=1}^{r_n}, \|z_n\|\} = 2^{-r_1 - \cdots - r_{n-1}} \leq 2^{-n}. \quad (10)$$

By the equation (8), we get

$$\|b_n\| \leq 2 \cdot 2^{-2n} \cdot \left\| \sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)} - 1} e_{i,i+1}^{(n,s)} \right\| \leq 2^{-2n+1} \leq 2^{-n}. \quad (11)$$

It induces that both  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are all convergent series in  $\mathcal{A}$ . Let

$$a = \sum_{n=1}^{\infty} a_n, \quad b = \sum_{n=1}^{\infty} b_n. \quad (12)$$

It is clear that  $a = a^* \in \mathcal{A}$  and  $b = b^* \in \mathcal{A}$ .

CLAIM 3.3. *Let  $\{\mathcal{B}_n\}_{n=1}^{\infty}$ ,  $\{z_n\}_{n=1}^{\infty}$  and  $a, b$  be defined as above. Then*

$$\{\mathcal{B}_1, z_1, \mathcal{B}_2, z_2, \dots\} \subseteq C^*(a, b),$$

where  $C^*(a, b)$  is the  $C^*$ -subalgebra generated by  $a$  and  $b$  in  $\mathcal{A}$ .

*Proof of the claim.* It is sufficient to prove that, for any  $n \geq 1$ ,

$$\{\mathcal{B}_1, \dots, \mathcal{B}_n, z_1, \dots, z_n\} \subseteq C^*(a, b).$$

We will prove it by using the induction on  $n$ .

First we prove  $\{\mathcal{B}_1, z_1\} \subseteq C^*(a, b)$ . From equations (7), (9), (12) and part (i) of Claim 3.2, it follows that

$$\begin{aligned} \forall k \in \mathbb{N}, \quad (2a)^k &= (2a_1)^k + \sum_{n=2}^{\infty} (2a_n)^k \\ &= e_{11}^{(1,1)} + \sum_{s_1=2}^{r_1} (2^{-s_1+1})^k e_{11}^{(1,s_1)} + (2z_1)^k + \sum_{n=2}^{\infty} (2a_n)^k. \end{aligned}$$



By inequalities (6), (9) and (10), we obtain that

$$\left\| \sum_{s_1=2}^{r_1} (2^{-s_1+1})^k e_{11}^{(1,s_1)} + (2z_1)^k + \sum_{n=2}^{\infty} (2a_n)^k \right\| \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

which implies  $\|(2a)^k - e_{11}^{(1,1)}\| \rightarrow 0$ , as  $k$  goes to  $\infty$ . Thus,  $e_{11}^{(1,1)} \in C^*(a, b)$ . By the construction of the element  $b$ , it is not hard to check  $\{e_{ij}^{(1,1)} : 1 \leq i, j \leq k_1^{(1)}\}$  are contained in the  $C^*$ -subalgebra generated by  $e_{11}^{(1,1)}$  and  $b$  in  $\mathcal{A}$ . Therefore,  $\{e_{ij}^{(1,1)} : 1 \leq i, j \leq k_1^{(1)}\} \subseteq C^*(a, b)$ .

Since

$$(I_{\mathcal{A}} - e_{11}^{(1,1)}) \cdot a \cdot (I_{\mathcal{A}} - e_{11}^{(1,1)}) = \sum_{s=2}^{r_1} 2^{-s} e_{11}^{(1,s)} + z_1 + \sum_{n=2}^{\infty} a_n,$$

by equation (6) and inequality (10) we know

$$\left\| \left( 4(I_{\mathcal{A}} - e_{11}^{(1,1)}) \cdot a \cdot (I_{\mathcal{A}} - e_{11}^{(1,1)}) \right)^k - e_{11}^{(1,2)} \right\| \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

This implies  $e_{11}^{(1,2)}$  is in  $C^*(a, b)$ , whence  $\{e_{ij}^{(1,2)} : 1 \leq i, j \leq k_2^{(1)}\}$  are in  $C^*(a, b)$ . Repeating the preceding process, we get that  $\{e_{ij}^{(1,s)} : 1 \leq i, j \leq k_s^{(1)}, 1 \leq s \leq r_1\}$  and, therefore  $p_1$ , are contained in  $C^*(a, b)$ . By fact (i) of the Claim 3.2, we know

$$(I_{\mathcal{A}} - p_1)z_1 = z_1;$$

and, from equation (7) we know

$$(I_{\mathcal{A}} - p_1) \sum_{n=2}^{\infty} a_n = 0.$$

This induces that  $z_1$  is also contained in  $C^*(a, b)$ . Now we conclude that both  $\mathcal{B}_1$  and  $z_1$  are contained in  $C^*(a, b)$ .

Assume that  $\{\mathcal{B}_1, \dots, \mathcal{B}_{n-1}, z_1, \dots, z_{n-1}\} \subseteq C^*(a, b)$ . We need to prove that  $\{\mathcal{B}_n, z_n\} \subseteq C^*(a, b)$ . By the equation (2) and the construction of the elements  $a, b$  (see equations (7), (8), (12)), we know that

$$(p_1 \cdots p_{n-1})a = \sum_{i=n}^{\infty} a_i = p_1 \cdots p_{n-1} \sum_{s=1}^{r_n} 2^{-r_1 - \cdots - r_{n-1} - s} \cdot e_{11}^{(n,s)} + z_n + \sum_{i=n+1}^{\infty} a_i,$$

and

$$(p_1 \cdots p_{n-1})b(p_1 \cdots p_{n-1}) = \sum_{i=n}^{\infty} b_i = \left( 2^{-2n} p_1 \cdots p_{n-1} \sum_{s=1}^{r_n} \sum_{i=1}^{k_s^{(n)}-1} (e_{i,i+1}^{(n,s)} + e_{i+1,i}^{(n,s)}) \right) + \sum_{i=n+1}^{\infty} b_i.$$

By equations (7), (9) and part (i) of Claim 3.2, we obtain that

$$\left\| \left( 2^{r_1 + \dots + r_{n-1} + 1} \left( \sum_{i=n}^{\infty} a_i \right)^k - (p_1 \cdots p_{n-1}) e_{11}^{(n,1)} \right) \right\| \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Using the similar argument as in the case  $n = 1$ , we know that  $\{p_1 \cdots p_{n-1} e_{ij}^{(n,s)} : 1 \leq i, j \leq k_s^{(n)}, 1 \leq s \leq r_n\}$  are contained in the C\*-subalgebra generated by  $\sum_{i=n}^{\infty} a_i$  and  $\sum_{i=n}^{\infty} b_i$  in  $\mathcal{A}$ . From the fact that  $\mathcal{B}_1, \dots, \mathcal{B}_n$  are mutually commuting subalgebras, it follows that

$$\begin{aligned} e_{ij}^{(n,s)} &= \sum_{s_1=1}^{r_1} \sum_{i_1=1}^{k_{s_1}^1} \cdots \sum_{s_{n-1}=1}^{r_{n-1}} \sum_{i_{n-1}=1}^{k_{s_{n-1}}^{(n-1)}} e_{i_{n-1}, k_{s_{n-1}}^{(n-1)}}^{(n-1, s_{n-1})} \\ &\quad \cdots e_{i_1, k_{s_1}^{(1)}}^{(1, s_1)} \cdot (p_1 \cdots p_{n-1} e_{ij}^{(n,s)}) \cdot e_{k_{s_1}^{(1)}, i_1}^{(1, s_1)} \cdots e_{k_{s_{n-1}}^{(n-1)}, i_{n-1}}^{(n-1, s_{n-1})} \end{aligned}$$

is in  $C^*(a, b)$ , which implies that  $\{e_{ij}^{(n,s)} : 1 \leq i, j \leq k_s^{(n)}, 1 \leq s \leq r_n\}$ , therefore  $\mathcal{B}_n, p_n$  and  $z_n$ , are contained in  $C^*(a, b)$ . *This completes the proof of the claim.*

By Claim 3.2 and Claim 3.3,  $\mathcal{A}$  is generated by two self-adjoint elements  $a$  and  $b$ . Therefore  $\mathcal{A}$  is singly generated.  $\square$

Suppose  $\mathcal{A}$  is a unital separable C\*-algebra and  $\mathcal{B}$  is a UHF algebra. It is clear that  $\mathcal{A} \otimes \mathcal{B}$  is approximately divisible. Therefore Theorem 9 in [9] is a corollary of our theorem.

**COROLLARY 3.1.** *If  $\mathcal{A}$  is a unital separable C\*-algebra and  $\mathcal{B}$  is a UHF algebra, then  $\mathcal{A} \otimes \mathcal{B}$  is singly generated.*

#### 4. Topological free entropy dimension

In this section we show that the topological free entropy dimension of any finite generating set of a unital separable approximately divisible C\*-algebra is less than or equal to 1.

**4.1. Preliminaries.** We are going to recall Voiculescu's definition of topological free entropy dimension of an  $n$ -tuple of self-adjoint elements in a unital C\*-algebra.

For any element  $(A_1, \dots, A_n)$  in  $\mathcal{M}_k(\mathbb{C})^n$ , define the operator norm on  $\mathcal{M}_k(\mathbb{C})^n$  by

$$\|(A_1, \dots, A_n)\| = \max\{\|A_1\|, \dots, \|A_n\|\}.$$

For every  $\omega > 0$ , we define the  $\omega$ - $\|\cdot\|$ -ball  $Ball(B_1, \dots, B_n; \omega, \|\cdot\|)$  centered at  $(B_1, \dots, B_n)$  in  $\mathcal{M}_k(\mathbb{C})^n$  to be the subset of  $\mathcal{M}_k(\mathbb{C})^n$  consisting of all  $(A_1, \dots, A_n)$  in  $\mathcal{M}_k(\mathbb{C})^n$  such that

$$\|(A_1, \dots, A_n) - (B_1, \dots, B_n)\| < \omega.$$

Suppose  $\mathcal{F}$  is a subset of  $\mathcal{M}_k(\mathbb{C})^n$ . We define the *covering number*  $\nu_{\infty}(\mathcal{F}, \omega)$  to be the minimal number of  $\omega$ - $\|\cdot\|$ -balls whose union covers  $\mathcal{F}$  in  $\mathcal{M}_k(\mathbb{C})^n$ .

Define  $\mathbb{C}\langle X_1, \dots, X_n \rangle$  to be the unital noncommutative polynomials in the indeterminates  $X_1, \dots, X_n$ . Let  $\{p_m\}_{m=1}^\infty$  be the collection of all noncommutative polynomials in  $\mathbb{C}\langle X_1, \dots, X_n \rangle$  with rational complex coefficients. (Here “rational complex coefficients” means that the real and imaginary parts of all coefficients of  $p_m$  are rational numbers).

Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra,  $x_1, \dots, x_n, y_1, \dots, y_t$  are self-adjoint elements of  $\mathcal{A}$ . For any  $\omega, \varepsilon > 0$ , positive integers  $k$  and  $m$ , define

$$\Gamma_{top}(x_1, \dots, x_n; k, \varepsilon, m) = \{(A_1, \dots, A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n : \\ |||p_j(A_1, \dots, A_n)|| - ||p_j(x_1, \dots, x_n)||| < \varepsilon, \forall 1 \leq j \leq m\},$$

and define

$$\nu_\infty(\Gamma_{top}(x_1, \dots, x_n; k, \varepsilon, m), \omega)$$

to be the covering number of the set  $\Gamma_{top}(x_1, \dots, x_n; k, \varepsilon, m)$  by  $\omega$ - $\|\cdot\|$ -balls in the metric space  $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$  equipped with operator norm.

Define

$$\delta_{top}(x_1, \dots, x_n; \omega) = \inf_{\varepsilon > 0, m \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_{top}(x_1, \dots, x_n; k, \varepsilon, m), \omega))}{-k^2 \log \omega},$$

and

$$\delta_{top}(x_1, \dots, x_n) = \limsup_{\omega \rightarrow 0^+} \delta_{top}(x_1, \dots, x_n; \omega).$$

Define  $\Gamma_{top}(x_1, \dots, x_n : y_1, \dots, y_t; k, \varepsilon, m)$  to be the set of  $(A_1, \dots, A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n$  such that there is  $(B_1, \dots, B_t) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^t$  satisfying

$$(A_1, \dots, A_n, B_1, \dots, B_t) \in \Gamma_{top}(x_1, \dots, x_n, y_1, \dots, y_t; k, \varepsilon, m).$$

Then similarly we can define

$$\delta_{top}(x_1, \dots, x_n : y_1, \dots, y_t; \omega) = \inf_{\varepsilon > 0, m \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_{top}(x_1, \dots, x_n : y_1, \dots, y_t; k, \varepsilon, m), \omega))}{-k^2 \log \omega},$$

and

$$\delta_{top}(x_1, \dots, x_n : y_1, \dots, y_t) = \limsup_{\omega \rightarrow 0^+} \delta_{top}(x_1, \dots, x_n : y_1, \dots, y_t; \omega).$$

**LEMMA 4.1.** *Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra,  $x_1, \dots, x_n, y_1, \dots, y_t$  are self-adjoint elements in  $\mathcal{A}$  and  $x_1, \dots, x_n$  generate  $\mathcal{A}$ . Suppose  $p \in \{p_m\}_{m=1}^\infty$  and  $\omega > 0$ . Then the following are true:*

- (1)  $\delta_{top}(x_1, \dots, x_n; \omega) = \delta_{top}(x_1, \dots, x_n : y_1, \dots, y_t; \omega)$ ,
- (2)  $\delta_{top}(p(x_1, \dots, x_n) : x_1, \dots, x_n; \omega) = \delta_{top}(p(x_1, \dots, x_n) : x_1, \dots, x_n, y_1, \dots, y_t; \omega)$ ,
- (3)  $\delta_{top}(x_1, \dots, x_n) \geq \delta_{top}(p(x_1, \dots, x_n) : x_1, \dots, x_n)$ .

**Proof.** The proof of (1) and (2) are straightforward adaptations of the proof of Proposition 1.6 in [16]. (3) is proved by D. Hadwin and J. Shen in [7].  $\square$

The following lemma is Lemma 2.3 in [1], and it will be used in the proofs of Theorem 4.1 and Theorem 4.3.

LEMMA 4.2. *Let  $\mathcal{B}$  be a finite-dimensional  $C^*$ -algebra, which is isomorphic to  $\mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C})$ . For any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that, whenever  $\mathcal{A}$  is a unital separable  $C^*$ -algebra with the unit  $I_{\mathcal{A}}$  and  $\{a_{ij}^{(s)} : 1 \leq i, j \leq k_s, 1 \leq s \leq r\}$  in  $\mathcal{A}$  satisfying*

$$(1) \quad \|(a_{ij}^{(s)})^* - a_{ji}^{(s)}\| \leq \delta \text{ for all } i, j, s,$$

$$(2) \quad \|\sum_{s=1}^r \sum_{i=1}^{k_s} a_{ii}^{(s)} - I_{\mathcal{A}}\| \leq \delta,$$

$$(3) \quad \|a_{ij}^{(s)} a_{j_1}^{(s)} - a_{ij_1}^{(s)}\| \leq \delta \text{ for all } i, j, j_1, s, \|a_{ij}^{(s)} a_{i_1 j_1}^{(s_1)}\| \leq \delta \text{ if } s \neq s_1 \text{ or } j \neq i_1,$$

*then there is a set  $\{e_{ij}^{(s)} : 1 \leq i, j \leq k_s, 1 \leq s \leq r\}$  of matrix units for a copy of  $\mathcal{B}$  in  $\mathcal{A}$  satisfying  $\|a_{ij}^{(s)} - e_{ij}^{(s)}\| < \varepsilon$  for all  $i, j, s$ .*

#### 4.2. Upper bound of topological free entropy dimension in an approximately divisible $C^*$ -algebra.

The following lemma is Lemma 6 in [5].

LEMMA 4.3. *The following statements are true:*

(1) *Let  $\mathcal{U}_k$  be the group of all unitary matrices in  $\mathcal{M}_k(\mathbb{C})$ ,  $\omega > 0$ . Then*

$$\left(\frac{1}{\omega}\right)^{k^2} \leq \nu_{\infty}(\mathcal{U}_k, \omega) \leq \left(\frac{9\pi e}{\omega}\right)^{k^2}.$$

(2) *If  $d$  is a metric on  $\mathbb{R}^m$ ,  $\mathbb{B}$  is the unit ball of  $\mathbb{R}^m$  equipped with the norm induced by  $d$ , then for  $\omega > 0$ ,*

$$\left(\frac{1}{\omega}\right)^m \leq \nu_d(\mathbb{B}, \omega) \leq \left(\frac{3}{\omega}\right)^m$$

Let  $\mathcal{B}$  be a finite-dimensional  $C^*$ -algebra which is isomorphic to  $\mathcal{M}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{k_r}(\mathbb{C})$  for some positive integers  $k_1, \dots, k_r$ . To simplify the notation, we will use  $\{e_{st}^{(\iota)}\}_{s,t,\iota}$  to denote a set  $\{e_{st}^{(\iota)} : 1 \leq s, t \leq k_{\iota}, 1 \leq \iota \leq r\}$  of matrix units for  $\mathcal{B}$ , let  $\{\text{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}$  denote the set  $\{\frac{e_{st}^{(\iota)} + (e_{st}^{(\iota)})^*}{2} : 1 \leq s, t \leq k_{\iota}, 1 \leq \iota \leq r\}$ , and let  $\{\text{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}$  denote the set  $\{\frac{e_{st}^{(\iota)} - (e_{st}^{(\iota)})^*}{2\sqrt{-1}} : 1 \leq s, t \leq k_{\iota}, 1 \leq \iota \leq r\}$ .

LEMMA 4.4. *Let  $\mathcal{A}$  be a unital separable approximately divisible  $C^*$ -algebra with unit  $I_{\mathcal{A}}$ , and  $\{x_1, \dots, x_n\}$  be a family of self-adjoint generators of  $\mathcal{A}$ . Then, for any  $\omega > 0$  and positive integer  $N$ , there exists a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B} \subseteq \mathcal{A}$  with a set of matrix units  $\{e_{st}^{(\iota)}\}_{s,t,\iota} = \{e_{st}^{(\iota)} : 1 \leq s, t \leq k_{\iota}, 1 \leq \iota \leq r\}$ , a positive integer  $m_0$  and  $1 > \varepsilon_0 > 0$ , such that*

$$(1) \quad I_{\mathcal{A}} \in \mathcal{B},$$

$$(2) \quad \text{SubRank}(\mathcal{B}) \geq N,$$

$$(3) \quad \text{for any } m \geq m_0, \varepsilon \leq \varepsilon_0, \text{ and any } k \geq 1, \text{ if}$$

$$(A_1, \dots, A_n, \{B_{st}^{(\iota)}\}_{s,t,\iota}, \{C_{st}^{(\iota)}\}_{s,t,\iota}) \in \Gamma_{\text{top}}(x_1, \dots, x_n, \{\text{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\text{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m),$$

then there exists a set  $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$  of matrix units for a copy of  $\mathcal{B}$  in  $\mathcal{M}_k(\mathbb{C})$  so that

$$\|A_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} P_{ss}^{(\iota)} A_j P_{ss}^{(\iota)}\| \leq 2\omega.$$

PROOF. Suppose  $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$ , where  $\mathcal{A}_m$  is as in Proposition 2.1. For any  $\omega > 0$ , any positive integer  $N$  and self-adjoint elements  $x_1, \dots, x_n$ , there are self-adjoint elements  $y_1, \dots, y_n$  in some  $\mathcal{A}_m$  such that  $\|x_j - y_j\| < \frac{\omega}{2}$  for all  $1 \leq j \leq n$ . From part (3) of Proposition 2.1, there exists a finite-dimensional subalgebra  $\mathcal{B}$  of  $\mathcal{A}'_m \cap \mathcal{A}$  such that  $I_{\mathcal{A}} \in \mathcal{B}$  and  $\text{SubRank}(\mathcal{B}) \geq N$ . Let  $\{e_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$  be a set of matrix units for  $\mathcal{B}$ . Then, for  $1 \leq j \leq n$ ,

$$\begin{aligned} & \|x_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} e_{ss}^{(\iota)} x_j e_{ss}^{(\iota)}\| \\ &= \|(x_j - y_j) + y_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} e_{ss}^{(\iota)} (x_j - y_j) e_{ss}^{(\iota)} - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} e_{ss}^{(\iota)} y_j e_{ss}^{(\iota)}\| \\ &= \|(x_j - y_j) - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} e_{ss}^{(\iota)} (x_j - y_j) e_{ss}^{(\iota)}\| \\ &\leq \|x_j - y_j\| + \|\sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} e_{ss}^{(\iota)} (x_j - y_j) e_{ss}^{(\iota)}\| \\ &\leq \|x_j - y_j\| + \max\{\|e_{ss}^{(\iota)} (x_j - y_j) e_{ss}^{(\iota)}\|\} \\ &< \frac{\omega}{2} + \frac{\omega}{2} = \omega. \end{aligned}$$

Let  $R = \max\{\|x_1\|, \dots, \|x_n\|, 1\}$ .

By Lemma 4.2, there are  $0 < \varepsilon_0 < \min\{1, \frac{\omega}{2}\}$  and positive integer  $m_0$ , such that, for any  $m \geq m_0$ ,  $\varepsilon \leq \varepsilon_0$  and  $k \geq 1$ , if

$$(A_1, \dots, A_n, \{B_{st}^{(\iota)}\}_{s,t,\iota}, \{C_{st}^{(\iota)}\}_{s,t,\iota}) \in \Gamma_{top}(x_1, \dots, x_n, \{\text{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\text{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m),$$

then there exists a set  $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\} \subset \mathcal{M}_k^{s.a.}(\mathbb{C})$  such that

- (a)  $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$  is exactly a set of matrix units for a copy of  $\mathcal{B}$  in  $\mathcal{M}_k(\mathbb{C})$ ,
- (b) For any  $1 \leq \iota \leq r, 1 \leq s, t \leq k_\iota$ ,

$$\|P_{st}^{(\iota)} - (B_{st}^{(\iota)} + \sqrt{-1} \cdot C_{st}^{(\iota)})\| < \frac{\omega}{24R \cdot \text{Rank}(\mathcal{B})}.$$

Let  $D_{st}^{(\iota)} = B_{st}^{(\iota)} + \sqrt{-1} \cdot C_{st}^{(\iota)}$ . We have

$$\begin{aligned}
& \|A_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} P_{ss}^{(\iota)} A_j P_{ss}^{(\iota)}\| \\
& \leq \|A_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} D_{ss}^{(\iota)} A_j D_{ss}^{(\iota)}\| + \left\| \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) A_j D_{ss}^{(\iota)} \right\| \\
& \quad + \left\| \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} D_{ss}^{(\iota)} A_j (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) \right\| \\
& \quad + \left\| \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) A_j (P_{ss}^{(\iota)} - D_{ss}^{(\iota)}) \right\| \\
& \leq \omega + \varepsilon + \frac{\omega}{6} + \frac{\omega}{6} + \frac{\omega}{6} \\
& \leq 2\omega.
\end{aligned}$$

□

**THEOREM 4.1.** *Suppose  $\mathcal{A}$  is a unital separable approximately divisible  $C^*$ -algebra generated by self-adjoint elements  $x_1, \dots, x_n$ . Then*

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

**PROOF.** For any positive integer  $N$ ,  $1 > \omega > 0$ , from Lemma 4.4, there exists a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B} \subseteq \mathcal{A}$  with a set of matrix units  $\{e_{st}^{(\iota)}\}_{s,t,\iota} = \{e_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$ , a positive integer  $m_0$  and  $1 > \varepsilon_0 > 0$ , such that

- (a)  $I_{\mathcal{A}} \in \mathcal{B}$ , where  $I_{\mathcal{A}}$  is the unit of  $\mathcal{A}$ ,
- (b)  $\text{SubRank}(\mathcal{B}) \geq N$ ,
- (c) for  $m \geq m_0$  and  $\varepsilon \leq \varepsilon_0$ , and for any  $k \geq 1$ , if

$$(A_1, \dots, A_n, \{B_{st}^{(\iota)}\}_{s,t,\iota}, \{C_{st}^{(\iota)}\}_{s,t,\iota}) \in \Gamma_{top}(x_1, \dots, x_n, \{\text{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\text{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m), \quad (13)$$

then there exists a set  $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$  of matrix units for a copy of  $\mathcal{B}$  in  $\mathcal{M}_k(\mathbb{C})$  so that

$$\|A_j - \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} P_{ss}^{(\iota)} A_j P_{ss}^{(\iota)}\| \leq 2\omega.$$

Note that  $\{P_{ss}^{(\iota)} : 1 \leq \iota \leq r, 1 \leq s \leq k_\iota\}$  is a family of mutually orthogonal projections with the sum  $I_k$  in  $\mathcal{M}_k(\mathbb{C})$ . There is some unitary matrix  $U \in \mathcal{U}_k$  such that  $U^* P_{ss}^{(\iota)} U (= Q_{ss}^{(\iota)})$  is diagonal for any  $1 \leq \iota \leq r$  and  $1 \leq s \leq k_\iota$ . Then, for any  $1 \leq j \leq n$ ,

$$\|A_j - U \left( \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} (U^* A_j U) Q_{ss}^{(\iota)} \right) U^*\| \leq 2\omega. \quad (14)$$

Thus, for  $1 \leq j \leq n$ ,

$$\left\| \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} (U^* A_j U) Q_{ss}^{(\iota)} \right\| \leq \|A_j\| + 2\omega \leq 4R.$$

Therefore

$$\left( \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} (U^* A_1 U) Q_{ss}^{(\iota)}, \dots, \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} (U^* A_n U) Q_{ss}^{(\iota)} \right) \in \text{Ball}(0, \dots, 0; 4R, \|\cdot\|), \quad (15)$$

i.e., it is contained in the ball centered at  $(0, \dots, 0)$  with radius  $4R$  in  $(\mathcal{M}_k(\mathbb{C}))^n$ .

Since  $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$  is a system of matrix units for a copy of  $\mathcal{B}$  in  $\mathcal{M}_k(\mathbb{C})$  such that

$$\sum_{1 \leq \iota \leq r, 1 \leq s \leq k_\iota} P_{ss}^{(\iota)} = I_k,$$

we know that there is a unital embedding from  $\mathcal{B}$  into  $\mathcal{M}_k(\mathbb{C})$ . It follows that there are positive integers  $c_1, \dots, c_r$  satisfying

- (i)  $\text{Rank } P_{11}^{(\iota)} = \dots = \text{Rank } P_{k_\iota, k_\iota}^{(\iota)} = c_\iota$  for all  $1 \leq \iota \leq r$ , where  $\text{Rank } T$  is the rank of the matrix  $T$  for any  $T$  in  $\mathcal{M}_k(\mathbb{C})$ ; and
- (ii)  $c_1 k_1 + \dots + c_r k_r = k$ .

By the restriction on the  $C^*$ -algebra  $\mathcal{B}$  (see condition (b) as above), we know that  $\text{SubRank}(\mathcal{B}) \geq N$ , i.e.,

$$\min\{k_1, \dots, k_r\} \geq N.$$

By (ii), we obtain that

$$\min\{c_1, \dots, c_r\} \leq \frac{k}{N}. \quad (16)$$

By (i), we know that

$$\text{Rank } Q_{11}^{(\iota)} = \dots = \text{Rank } Q_{k_\iota, k_\iota}^{(\iota)} = \text{Rank } P_{11}^{(\iota)} = \dots = \text{Rank } P_{k_\iota, k_\iota}^{(\iota)} = c_\iota, \quad \text{for } 1 \leq \iota \leq r.$$

Thus the real-dimension of the linear space  $\sum_{1 \leq \iota \leq r} \sum_{1 \leq j \leq k_\iota} Q_{jj}^{(\iota)} \mathcal{M}_k(\mathbb{C})^{s.a} Q_{jj}^{(\iota)}$  is

$$\dim_{\mathbb{R}} \left( \sum_{1 \leq \iota \leq r} \sum_{1 \leq j \leq k_\iota} Q_{jj}^{(\iota)} \mathcal{M}_k(\mathbb{C})^{s.a} Q_{jj}^{(\iota)} \right) = c_1^2 k_1 + \dots + c_r^2 k_r. \quad (17)$$

By the inequality (16), we get

$$c_1^2 k_1 + \dots + c_r^2 k_r \leq \frac{k}{N} (c_1 k_1 + \dots + c_r k_r) = \frac{k^2}{N}. \quad (18)$$

For any such family of positive integers  $c_1, \dots, c_r$  with  $c_1 k_1 + \dots + c_r k_r = k$ , and the family of mutually orthogonal diagonal projections  $\{Q_{ss}^{(\iota)}\}_{1 \leq s \leq k_\iota, 1 \leq \iota \leq r}$  with

$$\text{Rank}(Q_{ss}^{(\iota)}) = c_\iota, \quad \forall 1 \leq \iota \leq r,$$

we define

$$\begin{aligned} \Omega(\{Q_{ss}^{(\iota)}\}_{s, \iota}) &= \left\{ \left( \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} T_1 Q_{ss}^{(\iota)}, \dots, \sum_{1 \leq \iota \leq r} \sum_{1 \leq s \leq k_\iota} Q_{ss}^{(\iota)} T_n Q_{ss}^{(\iota)} \right) : \right. \\ &\quad \left. T_i = T_i^* \in \mathcal{M}_k(\mathbb{C}), \quad \forall 1 \leq i \leq n \right\}, \end{aligned}$$

which is a subset of  $(\mathcal{M}_k^{s.a.}(\mathbb{C}))^n$ .

Therefore, combining Lemma 4.3, equation (17) and inequality (18), for any  $\omega > 0$  we have

$$\nu_\infty(\Omega(\{Q_{ss}^{(\iota)}\}_{s,\iota}) \cap \text{Ball}(0, \dots, 0; 4R, \|\cdot\|); \omega) \leq \left(\frac{12R}{\omega}\right)^{\frac{nk^2}{N}}. \quad (19)$$

Let

$$\Lambda = \{(c_1, \dots, c_r) : \exists k_1, \dots, k_r \in \mathbb{N} \text{ such that } c_1 k_1 + \dots + c_r k_r = k\}.$$

By inequality (13), we know that the cardinality of the set  $\Lambda$  satisfies

$$\text{Card}(\Lambda) \leq \left(\frac{k}{N}\right)^r. \quad (20)$$

Let

$$\Omega = \cup_{(c_1, \dots, c_r) \in \Lambda} \left\{ \Omega(\{Q_{ss}^{(\iota)}\}_{s,\iota}) \mid \{Q_{ss}^{(\iota)}\}_{1 \leq s \leq k_\iota, 1 \leq \iota \leq r} \text{ is a family of mutually orthogonal diagonal projections with } \text{Rank}(Q_{ss}^{(\iota)}) = c_\iota, \forall 1 \leq \iota \leq r \right\}.$$

By inequalities (19) and (20), we know that

$$\nu_\infty(\Omega \cap \text{Ball}(0, \dots, 0; 4R, \|\cdot\|); \omega) \leq \left(\frac{12R}{\omega}\right)^{\frac{nk^2}{N}} \cdot \left(\frac{k}{N}\right)^r. \quad (21)$$

Based on (13), (14), (15), (21), and Lemma 4.3, now it is a standard argument to show:

$$\begin{aligned} & \delta_{top}(x_1, \dots, x_n : \{\text{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\text{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; 4\omega) \\ & \leq \limsup_{k \rightarrow \infty} \frac{\log \left( \left(\frac{k}{N}\right)^r \left(\frac{12R}{\omega}\right)^{\frac{nk^2}{N}} \left(\frac{9\pi e}{\omega}\right)^{k^2} \right)}{-k^2 \log(4\omega)} \\ & = 1 + \frac{n}{N} + \frac{\frac{n}{N} \log(12R) + \log(9\pi e)}{-\log \omega}. \end{aligned}$$

By Lemma 4.1,

$$\delta_{top}(x_1, \dots, x_n; 4\omega) \leq 1 + \frac{n}{N} + \frac{\frac{n}{N} \log(12R) + \log(9\pi e)}{-\log \omega}.$$

Therefore

$$\delta_{top}(x_1, \dots, x_n) = \limsup_{\omega \rightarrow 0^+} \delta_{top}(x_1, \dots, x_n; 4\omega) \leq 1 + \frac{n}{N}.$$

Since  $N$  is arbitrarily large,  $\delta_{top}(x_1, \dots, x_n) \leq 1$ . □



### 4.3. Lower bound of topological free entropy dimension of an approximately divisible $C^*$ -algebra.

The following definition is Definition 5.3 in [7].

DEFINITION 4.1. Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $x_1, \dots, x_n$  is a family of self-adjoint elements of  $\mathcal{A}$  that generates  $\mathcal{A}$  as a  $C^*$ -algebra. If for any  $m \in \mathbb{N}$ ,  $\varepsilon > 0$ , there is a sequence of positive integers  $k_1 < k_2 < \dots$  such that, for  $s \geq 1$ ,

$$\Gamma_{top}(x_1, \dots, x_n : y_1, \dots, y_t; k_s, \varepsilon, m) \neq \emptyset,$$

then  $\mathcal{A}$  is called having approximation property.

Using the idea in the proof of Lemma 5.4 in [7], we can prove the following lemma.

LEMMA 4.5. Suppose  $m_1, m_2, \dots, m_r$  is a family of positive integers with summation  $m$  and  $m_1, \dots, m_r \geq N$  for some positive integer  $N$ . Suppose  $k_1, \dots, k_m$  is a family of positive integers with summation  $k$  and for every  $1 \leq s \leq r$ ,  $k_{m_1+\dots+m_{s-1}+1} = \dots = k_{m_1+\dots+m_s}$  ( $m_0 = 0$ ). If  $A = A^* \in \mathcal{M}_k(\mathbb{C})$ , and for some  $U \in \mathcal{U}_k$ ,

$$\|A - U \begin{pmatrix} 1 \cdot I_{k_1} & 0 & \dots & 0 \\ 0 & 2 \cdot I_{k_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & m \cdot I_{k_m} \end{pmatrix} U^*\| \leq \frac{2}{N^3},$$

then, for any  $\omega > 0$ , we have

$$\nu_\infty(\Omega(A), \omega) \geq (8C_1\omega)^{-k^2} \left( \frac{2C}{\omega} \right)^{\frac{-50k^2}{N}},$$

for some constants  $C_1, C > 1$  independent of  $k, \omega$ , where

$$\Omega(A) = \{W^*AW : W \in \mathcal{U}_k\}.$$

Now we are ready to prove the main theorem in this subsection.

THEOREM 4.2. Let  $\mathcal{A}$  be a unital separable approximately divisible  $C^*$ -algebra generated by self-adjoint elements  $x_1, \dots, x_n$ . If  $\mathcal{A}$  has approximation property, then

$$\delta_{top}(x_1, \dots, x_n) \geq 1.$$

PROOF. For any positive integer  $N$ , by part (3) of Proposition 2.1, there is a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  containing the unit of  $\mathcal{A}$  with  $\text{SubRank}(\mathcal{B}) \geq N$ . Therefore there are positive integers  $r, k_1, \dots, k_r$  such that

$$\mathcal{B} \simeq \mathcal{M}_{k_1}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{k_r}(\mathbb{C}).$$

Let  $\{e_{st}^{(\iota)} : 1 \leq \iota \leq r, 1 \leq s, t \leq k_\iota\}$  be a system of matrix units for  $\mathcal{B}$ . Let

$$z_N = \sum_{\iota=1}^r \sum_{s=1}^{k_\iota} \left( s + \sum_{j=1}^{\iota-1} k_j \right) \cdot e_{ss}^{(\iota)}.$$

Note that  $\{p_m(x_1, \dots, x_n)\}_{m=1}^\infty$  is a norm-dense set in  $\mathcal{A}$ . There exists a polynomial  $p_{m_N} \in \{p_m\}_{m=1}^\infty$  such that  $p_{m_N}(x_1, \dots, x_n)$  is self-adjoint and  $\|p_{m_N}(x_1, \dots, x_n) - z_N\| \leq \frac{1}{N^3}$ .

For sufficiently small  $\varepsilon > 0$ , sufficiently large positive integers  $m$  and  $k$ , if

$$(B, A_1, \dots, A_n, \{C_{st}^{(\iota)}\}_{s,t,\iota}, \{D_{st}^{(\iota)}\}_{s,t,\iota}) \\ \in \Gamma_{top}(p_{m_N}(x_1, \dots, x_n), x_1, \dots, x_n, \{\operatorname{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\operatorname{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m),$$

then, by Lemma 4.2, there exists a set  $\{P_{st}^{(\iota)} : 1 \leq s, t \leq k_\iota, 1 \leq \iota \leq r\}$  of matrix units for a copy of  $\mathcal{B}$  in  $\mathcal{M}_k(\mathbb{C})$ , such that

$$\|B - \sum_{\iota=1}^r \sum_{s=1}^{k_\iota} \left( s + \sum_{j=1}^{\iota-1} k_j \right) \cdot P_{ss}^{(\iota)}\| \leq \frac{2}{N^3}.$$

Let  $U$  be a unitary matrix in  $\mathcal{M}_k(\mathbb{C})$  such that, for any  $1 \leq s \leq k_\iota$  and  $1 \leq \iota \leq r$ ,  $U^* P_{ss}^{(\iota)} U (= Q_{ss}^{(\iota)})$  is diagonal. Then, from the preceding inequality,

$$\|B - U \left( \sum_{\iota=1}^r \sum_{s=1}^{k_\iota} \left( s + \sum_{j=1}^{\iota-1} k_j \right) \cdot Q_{ss}^{(\iota)} \right) U^*\| \leq \frac{2}{N^3}.$$

From Lemma 4.5, for any  $\omega > 0$ , when  $m$  is large enough and  $\varepsilon$  is small enough, there are some constants  $C, C_1 > 1$  independent of  $k$  and  $\omega$ , such that

$$\nu_\infty(\Gamma_{top}(p_{m_N}(x_1, \dots, x_n) : x_1, \dots, x_n, \{\operatorname{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\operatorname{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}; k, \varepsilon, m), \omega) \\ \geq (8C_1\omega)^{-k^2} \left( \frac{2C}{\omega} \right)^{\frac{-50k^2}{N}}.$$

Therefore

$$\delta_{top}(p_{m_N}(x_1, \dots, x_n) : x_1, \dots, x_n, \{\operatorname{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\operatorname{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}) \geq 1 - \frac{50}{N}.$$

By Lemma 4.1,

$$\begin{aligned} & \delta_{top}(p_{m_N}(x_1, \dots, x_n) : x_1, \dots, x_n, \{\operatorname{Re}(e_{st}^{(\iota)})\}_{s,t,\iota}, \{\operatorname{Im}(e_{st}^{(\iota)})\}_{s,t,\iota}) \\ &= \delta_{top}(p_{m_N}(x_1, \dots, x_n) : x_1, \dots, x_n) \\ &\leq \delta_{top}(x_1, \dots, x_n), \end{aligned}$$

whence  $\delta_{top}(x_1, \dots, x_n) \geq 1 - \frac{50}{N}$ . Since  $N$  is an arbitrary positive integer, we obtain

$$\delta_{top}(x_1, \dots, x_n) \geq 1.$$

□

Combining Theorem 3.1, Theorem 4.1 and Theorem 4.2, we have the following result.

**THEOREM 4.3.** *Let  $\mathcal{A}$  be a unital separable approximately divisible  $C^*$ -algebra. If  $\mathcal{A}$  has approximation property, then*

$$\delta_{top}(x_1, \dots, x_n) = 1,$$

where  $x_1, \dots, x_n$  is any family of self-adjoint generators of  $\mathcal{A}$ .

Using the similar idea in the proof of Theorem 4.3, we can have the following generalized theorem.

**THEOREM 4.4.** *Let  $\mathcal{A}$  be a unital separable  $C^*$ -algebra generated by self-adjoint elements  $x_1, \dots, x_n$ . Suppose, for any positive integer  $N$ , there is a finite-dimensional subalgebra in  $\mathcal{A}$  containing the unit of  $\mathcal{A}$  with subrank at least  $N$ . If  $\mathcal{A}$  has approximation property, then  $\delta_{top}(x_1, \dots, x_n) = 1$ .*

## 5. Similarity degree

In 1955, R. Kadison [8] formulated the following conjecture: Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and let  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  ( $\mathcal{H}$  is a Hilbert space) be a unital bounded homomorphism. Then  $\pi$  is similar to a  $*$ -homomorphism, that is, there exists an invertible operator  $S \in \mathcal{B}(\mathcal{H})$  such that  $S^{-1}\pi(\cdot)S$  is a  $*$ -homomorphism.

This conjecture remains unproved, although many partial results are known. U. Haagerup [6] proved that  $\pi$  is similar to a  $*$ -homomorphism if and only if it is completely bounded. Moreover,

$$\|\pi\|_{cb} = \inf \{ \|S\| \cdot \|S^{-1}\| \}$$

where the infimum runs over all invertible  $S$  such that  $S^{-1}\pi(\cdot)S$  is a  $*$ -homomorphism. By definition,  $\|\pi\|_{cb} = \sup_{n \geq 1} \|\pi_n\|$  where  $\pi_n : \mathcal{M}_n(\mathcal{A}) \rightarrow \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$  is the mapping taking  $n$  by  $n$  matrix  $[a_{ij}]_{n \times n}$  to matrix  $[\pi(a_{ij})]_{n \times n}$ .

G. Pisier [10] proved that if a unital  $C^*$ -algebra  $\mathcal{A}$  verifies Kadison's conjecture, then there is a number  $d$  for which there exists a constant  $K$  so that any bounded homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  satisfies  $\|\pi\|_{cb} \leq K\|\pi\|^d$ . Moreover, the smallest number  $d$  with the property is an integer denoted by  $d(\mathcal{A})$  and called *similarity degree*. It is clear that a  $C^*$ -algebra  $\mathcal{A}$  verifies Kadison's conjecture if and only if  $d(\mathcal{A}) < \infty$ .

**REMARK 5.1.** *When determining  $d(\mathcal{A})$ , it is only necessary to consider unital bounded homomorphisms that are one-to-one. To see this, let  $\pi_0$  be a unital  $*$ -isomorphism from  $\mathcal{A}$  to  $\mathcal{B}(\mathcal{K})$  for some Hilbert space  $\mathcal{K}$ . It is not difficult to see that  $\pi \oplus \pi_0$  is one-to-one,  $\|\pi \oplus \pi_0\| = \|\pi\|$  and  $\|\pi \oplus \pi_0\|_{cb} = \|\pi\|_{cb}$ .*

We will show that the similarity degree of every unital separable approximately divisible  $C^*$ -algebra is at most 5. To do that, we need the following lemma.

**LEMMA 5.1.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra with the unit  $I_{\mathcal{A}}$ ,  $\mathcal{A}_0$  and  $\mathcal{B}$  be commuting  $C^*$ -subalgebras of  $\mathcal{A}$  that contain  $I_{\mathcal{A}}$ . Suppose  $\mathcal{B} \cong \mathcal{M}_{k_1}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{k_r}(\mathbb{C})$  with  $k_1, \dots, k_r \geq$*

$n \geq 2$  for some positive integer  $n$ , and  $\{e_{ij}^{(s)} : 1 \leq i, j \leq k_s, 1 \leq s \leq r\}$  is a set of matrix units for  $\mathcal{B}$ . If  $\{a_{ij} : 1 \leq i, j \leq n\} \subseteq \mathcal{A}_0$ , then

$$\left\| \sum_{1 \leq s \leq r} \sum_{1 \leq i, j \leq n} a_{ij} e_{ij}^{(s)} \right\| = \|[a_{ij}]_{n \times n}\|.$$

PROOF. Let  $p_1 = I_{k_1} \oplus \cdots \oplus 0 \oplus 0, \dots, p_r = 0 \oplus \cdots \oplus 0 \oplus I_{k_r}$  be the projections in  $\mathcal{B}$ , where  $I_{k_s}$  is the unit of  $\mathcal{M}_{k_s}(\mathbb{C})$  ( $1 \leq s \leq r$ ). Then it is clear that  $p_1 + \cdots + p_r = I_{\mathcal{A}}$  and for any  $1 \leq s \leq r$ ,  $1 \leq i, j \leq k_s$ ,  $e_{ij}^{(s)} = p_s e_{ij}^{(s)}$ .

Define

$$\pi : \mathcal{M}_{k_1}(p_1 \mathcal{A}_0) \oplus \cdots \oplus \mathcal{M}_{k_r}(p_r \mathcal{A}_0) \rightarrow C^*(\mathcal{A}_0, \mathcal{B})$$

by

$$\pi([p_1 a_{ij}^{(1)}]_{k_1 \times k_1} \oplus \cdots \oplus [p_r a_{ij}^{(r)}]_{k_r \times k_r}) = \sum_{s=1}^r \sum_{i,j=1}^{k_s} a_{ij}^{(s)} e_{ij}^{(s)},$$

for any  $a_{ij}^{(s)} \in \mathcal{A}_0$ . It is clear that  $\pi$  is a \*-isomorphism.

Thus, in  $\mathcal{M}_n(\mathcal{A})$ ,

$$\begin{aligned} \|[a_{ij}]_{n \times n}\| &= \left\| \sum_{s=1}^r \begin{pmatrix} p_s & & 0 \\ & \ddots & \\ 0 & & p_s \end{pmatrix} [a_{ij}]_{n \times n} \right\| \\ &= \max\{\|[p_s a_{ij}]_{n \times n}\| : 1 \leq s \leq n\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left\| \sum_{1 \leq s \leq r} \sum_{1 \leq i, j \leq n} a_{ij} e_{ij}^{(s)} \right\| \\ &= \left\| \pi \left( \begin{pmatrix} [p_1 a_{ij}]_{n \times n} & 0 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} [p_r a_{ij}]_{n \times n} & 0 \\ 0 & 0 \end{pmatrix} \right) \right\| \\ &= \max\{\|[p_s a_{ij}]_{n \times n}\| : 1 \leq s \leq r\}. \end{aligned}$$

□

THEOREM 5.1. If  $\mathcal{A}$  is a unital separable approximately divisible  $C^*$ -algebra, then

$$d(\mathcal{A}) \leq 5.$$

PROOF. Let  $\mathcal{A} = \overline{\cup_m \mathcal{A}_m}^{\|\cdot\|}$  with  $\mathcal{A}_m$  defined in Proposition 2.1. By Remark 5.1, let  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  be a one-to-one unital bounded homomorphism, where  $\mathcal{H}$  is a Hilbert space. It is sufficient to prove that

$$\|\pi|_{\cup_m \mathcal{A}_m}\|_{cb} \leq K \|\pi\|^5$$

for some constant  $K$ .

For any positive integer  $n$ , let  $\{a_{ij} : 1 \leq i, j \leq n\}$  be a family of elements in  $\cup_m \mathcal{A}_m$ . Then there exists some positive integer  $m_0$  such that  $\{a_{ij} : 1 \leq i, j \leq n\}$  is in  $\mathcal{A}_{m_0}$ . From

Proposition 2.1, there exists a finite-dimensional  $C^*$ -subalgebra  $\mathcal{B}$  containing the unit of  $\mathcal{A}$  with  $\text{SubRank}(\mathcal{B}) \geq n$  and  $\mathcal{B} \subset \mathcal{A}'_{m_0} \cap \mathcal{A}$ . Let  $\{e_{ij}^{(s)} : 1 \leq i, j \leq k_s, 1 \leq s \leq r\}$  be a set of matrix units for  $\mathcal{B}$ .

Since  $\mathcal{B}$  is finite-dimensional, it follows that  $\mathcal{B}$  is nuclear. Therefore, from [6], there exists an invertible operator  $S$  in  $\mathcal{B}(\mathcal{H})$ , such that  $\|S\| \cdot \|S^{-1}\| \leq C\|\pi\|^2$  for some constant  $C$ , and  $S^{-1}\pi|_{\mathcal{B}}S$  is a  $*$ -isomorphism. Let  $\rho = S^{-1}\pi S$ . Then  $\{\rho(e_{ij}^{(s)}) : 1 \leq i, j \leq k_s, 1 \leq s \leq r\}$  is a set of matrix units for the  $C^*$ -algebra  $\rho(\mathcal{B})$ . Hence, by Lemma 5.1,

$$\|\rho(\sum_{1 \leq s \leq r} \sum_{1 \leq i, j \leq n} a_{ij} e_{ij}^{(s)})\| \leq \|\rho\| \cdot \|\sum_{1 \leq s \leq r} \sum_{1 \leq i, j \leq n} a_{ij} e_{ij}^{(s)}\| = \|\rho\| \cdot \|[a_{ij}]_{n \times n}\|.$$

On the other hand, by Lemma 5.1,

$$\begin{aligned} & \|\rho(\sum_{1 \leq s \leq r} \sum_{1 \leq i, j \leq n} a_{ij} e_{ij}^{(s)})\| \\ &= \|\sum_{s=1}^r \sum_{1 \leq i, j \leq n} \rho(a_{ij}) \rho(e_{ij}^{(s)})\| \\ &= \|\rho(a_{ij})\|_{n \times n}. \end{aligned}$$

Therefore we get

$$\|\rho(a_{ij})\|_{n \times n} \leq \|\rho\| \cdot \|[a_{ij}]_{n \times n}\| \leq \|S\| \cdot \|S^{-1}\| \cdot \|\pi\| \cdot \|[a_{ij}]_{n \times n}\| \leq C\|\pi\|^3 \|[a_{ij}]_{n \times n}\|,$$

which implies that  $\|\rho|_{\cup_m \mathcal{A}_m}\|_{cb} \leq C\|\pi\|^3$ , then

$$\|\pi|_{\cup_m \mathcal{A}_m}\|_{cb} = \|S\rho|_{\cup_m \mathcal{A}_m} S^{-1}\|_{cb} \leq \|S^{-1}\| \cdot \|S\| \cdot \|\rho|_{\cup_m \mathcal{A}_m}\|_{cb} \leq C^2 \|\pi\|^5.$$

□

F. Pop [14] proved that if  $\mathcal{A}$  is a unital  $C^*$ -algebra,  $\mathcal{B}$  is a unital nuclear  $C^*$ -algebra and contains unital matrix algebras of any order, then the similarity degree of  $\mathcal{A} \otimes \mathcal{B}$  is at most 5. Here we state our another result which generalize F. Pop's result.

To prove our result, we need the following lemma (Corollary 2.3 in [14]).

**LEMMA 5.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be unital  $C^*$ -algebras and  $\mathcal{B}$  nuclear. If  $\pi$  is a unital bounded homomorphism of  $\mathcal{A} \otimes \mathcal{B}$  such that  $\pi|_{\mathcal{A}}$  is completely bounded and  $\pi|_{\mathcal{B}}$  is  $*$ -homomorphism, then  $\pi$  is completely bounded and  $\|\pi\|_{cb} \leq \|\pi|_{\mathcal{A}}\|_{cb}$ .*

Using Lemma 5.2 and the idea in the proof of Theorem 5.1, we can get the following theorem:

**THEOREM 5.2.** *Let  $\mathcal{A}$  be a unital nuclear  $C^*$ -algebra such that for any positive integer  $N$ , there is a finite-dimensional subalgebra in  $\mathcal{A}$  containing the unit of  $\mathcal{A}$  with subrank at least  $N$ . Then, for any unital  $C^*$ -algebra  $\mathcal{B}$ ,  $d(\mathcal{A} \otimes \mathcal{B}) \leq 5$ .*

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